Springs

Any elastic member which can deform under a force can act as a spring. The main function of a spring is to deflect under a load and to recover the original shape when the load is released. However, the term is normally used for those members that deform considerably under the action of forces without exceeding the safe limit of stresses, e.g., a steel helical spring may be expanded to twice its length without losing its elasticity. Springs can be made to act under tension, compression, torsion, bending or a combination of these loads. Usually, the springs are made from conventional metals though sometimes they can be of nonmetallic materials.

In general, the springs are used to

— absorb energy and to release the same according to the desired function to be performed such as a spring of a clock

— absorb shocks as in case of automobiles

— deflect under external forces to provide the desired motion to a machine member such as springs used in weighing machines, safety valves, clutches governors, etc.

- *Stiffness* or *spring constant* of a spring is defined as the force required for unit deflection.
- *Solid length* of a spring is the length of a spring in the fully compressed state when the coils touch each other.

11.1 CLOSE-COILED HELICAL SPRINGS

Close-coiled helical springs are those in which the angle of the helix is so small that if the axis of the spring is vertical, the coils may be assumed to be in a horizontal plane. Such a spring may be acted upon by an axial load or an axial torque. When they are acted upon by an axial load, there is axial extension and when there is an axial torque, there is a change in the radius of curvature of the spring coils. In the latter case, there is an angular rotation of the free end and the action is known as *wind-up*.

(i) Under Axial Load

Let W = Axial load

D = Mean coil diameter

R = Mean coil radius

d = wire diameter

θ = total angle of twist along the wire

δ = deflection of W along the axis of the coils
n = number of coils
l = length of wire

As shown in Fig. 11.1, the action of the load W on any cross-section is to twist it like a shaft with a pure torque WR. Bending and shear effects may be neglected.

Equating the work done by the axial force to the torsional strain energy

i.e., $\frac{1}{2}W\delta = \frac{1}{2}\cdot T\theta = \frac{1}{2}\cdot WR\cdot\theta$

or $\delta = R\cdot\theta$

Now, $$\theta = \frac{Tl}{GJ} = \frac{WRl}{G(\pi d^4/32)} = \frac{32WRl}{G\pi d^4} \qquad (11.1)$$

Also as $l = 2\pi Rn$,

$$\therefore \quad \theta = \frac{32WR(2\pi Rn)}{G\pi d^4} = \frac{64WR^2 n}{Gd^4} \qquad (11.2)$$

Deflection of the spring, $$\delta = R\theta = \frac{32WR^2 l}{G\pi d^4} \qquad (11.3)$$

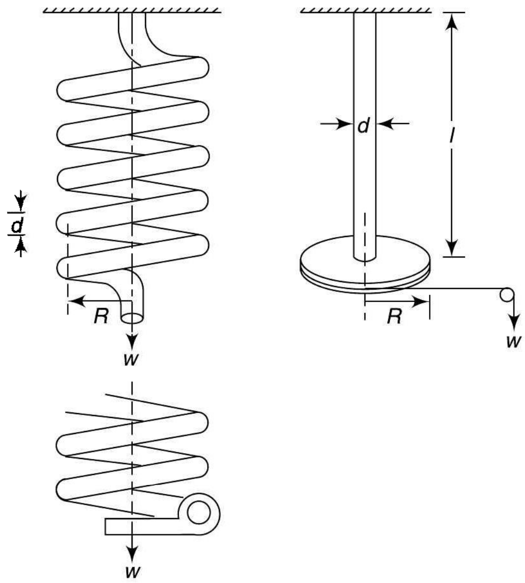

Fig. 11.1

Also by using Eq. 11.2, $$\delta = R\theta = \frac{64WR^3 n}{Gd^4} = \frac{8WD^3 n}{Gd^4} \qquad (11.4)$$

Stresses Direct shear stress $= \frac{W}{A} = \frac{W}{(\pi/4)d^2}$ (assuming uniform distribution)

Maximum torsional stress, $\tau = \frac{T}{J}\cdot r = \frac{WR}{(\pi d^4/32)}\cdot\frac{d}{2} = \frac{16WR}{\pi d^3} = \frac{8WD}{\pi d^3}$

$$\therefore \text{ maximum shear stress} = \frac{W}{(\pi/4)d^2} + \frac{8WD}{\pi d^3} = \frac{8WD}{\pi d^3}\left(\frac{d}{2D}+1\right) \qquad (11.5)$$

For most springs, $\frac{d}{2D} << 1$ and thus can be ignored.

$$\text{Thus shear stress} = \frac{8WD}{\pi d^3} \qquad (11.5a)$$

Equation 11.5a can be applied to springs of thin wires where D/d ratio is more than 20

Wahl's correction This is a method to incorporate the effects of direct shear and the curvature and is based on the experimental investigations of A H Wahl. According to him, the maximum stress in the spring wire can be expressed as

$$\tau_{max} = \frac{8WD}{\pi d^3}\cdot K \qquad (11.5b)$$

K is known as the *Wahl's constant* and is given by

$$K = \frac{4S_i - 1}{4S_i - 4} + \frac{0.615}{S_i} \qquad \text{(for inner surface)}$$

and $$K = \frac{4S_i - 1}{4S_i - 4} - \frac{0.615}{S_i}$$ (for outer surface)

where S_i is the *spring index* and is the ratio of the mean diameter of the coil to the diameter of the wire.

Equation 11.5b is suitable for heavy springs used for railway wagons.

(ii) Under Axial Torque

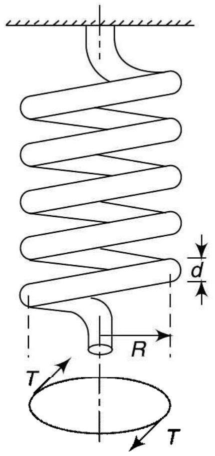

Fig. 11.2

The axial torque T tends to wind up the spring by producing approximately a pure bending moment at all cross-sections (Fig. 11.2).

When bending moment is applied to curved bars with small curvature (T is the applied bending moment),

$$T = EI\left(\frac{1}{R'} - \frac{1}{R}\right) \quad \text{(Eq. 9.1)}$$

where R and R' are the radii of curvature before and after applying the bending moment.

Let n and n' be the number of turns of the spring before and after applying the bending moment.

Then length of spring, $l = 2\pi nR = 2\pi n'R'$ or $n' = n(R/R')$

If φ is the angle of rotation of one end of the spring relative to the other end about the axis,

$$\varphi = 2\pi \times \text{increase in number of turns of the spring}$$

$$= 2\pi(n' - n) = 2\pi[n(R/R') - n] = 2\pi nR\left(\frac{1}{R'} - \frac{1}{R}\right) = \frac{Tl}{EI}$$

$$= \frac{Tl}{E(\pi d^4/64)} = \frac{64Tl}{E\pi d^4} = \frac{64T(\pi Dn)}{E\pi d^4} = \frac{64TDn}{Ed^4} \quad (11.6)$$

- The relation can also be obtained by considering the total strain energy due to bending moment of magnitude T,

  $$U = \frac{T^2 l}{2EI} \quad \text{(Eq. 7.34)}$$

  Then strain energy due to torsional effect,

  $$U = \frac{1}{2}T\varphi \quad \text{(Refer Section 10.6)}$$

  Equating the two, $\varphi = \frac{Tl}{EI} = \frac{64TDn}{Ed^4}$ as above

Example 11.1 A close-coiled helical spring having 24 turns is made of 8-mm diameter wire. The mean diameter of the spring is 80 mm and it carries a load of 250 N. Determine the shear stress developed, the deflection and the stiffness of the spring. Take G = 84 GPa.

Solution

Given A close-coiled helical spring

W = 250 N D = 80 mm

n = 24 d = 8 mm

G = 84 GPa

To find

Shear stress, deflection and stiffness

Shear stress

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 250 \times 80}{\pi \times 8^3} = 99.5 \text{ MPa}$$

Deflection

$$\delta = \frac{8WD^3 n}{Gd^4} = \frac{8 \times 250 \times 80^3 \times 24}{84\,000 \times 8^4} = 71.4 \text{ mm}$$

Stiffness

$$\text{Stiffness} = \frac{W}{\delta} = \frac{250}{71.4} = 3.5 \text{ N/mm}$$

Example 11.2 A close-coiled helical spring carries a load of 400 N. Its mean coil diameter is 10 times the wire diameter. Determine these diameters if the maximum value of shear stress in the spring is not to exceed 75 MPa.

Solution

Given A close-coiled helical spring

$W = 400$ N $\quad D = 10\,d$

$\tau = 75$ MPa

To find

— Wire diameter

— mean diameter

Wire diameter

$$\tau = \frac{8WD}{\pi d^3} \quad \text{or} \quad 75 = \frac{8 \times 400 \times (10d)}{\pi \times d^3} \quad \text{or} \quad d = 11.65 \text{ mm}$$

Mean diameter

$$D = 10 \times 11.65 = 116.5 \text{ mm}$$

Example 11.3 A safety valve of 120-mm diameter is designed to blow off at a gauge pressure of 1 MPa. A close-coiled helical spring of 160 mm mean diameter is used to hold the valve in position. Determine the diameter of the coils of the spring and the number of turns required if the initial compression of the spring is 60 mm and the maximum value of shear stress is 70 MPa. G = 84 GPa.

Solution

Given A close-coiled helical spring for a safety valve of 120-mm diameter with a blow-off pressure of 1 MPa.

$\delta = 60$ mm $\quad D = 160$ mm

$\tau = 70$ MPa $\quad G = 84$ GPa

To find

— Wire diameter

— Number of coils

Determination of load

$$\text{Force on the spring, } W = \frac{\pi}{4} \times (120)^2 \times 1 = 11\,310 \text{ N}$$

Wire diameter

$$\tau = \frac{8WD}{\pi d^3} \quad \text{or} \quad 70 = \frac{8 \times 11\,310 \times 200}{\pi \times d^3} \quad \text{or} \quad d = 43.5 \text{ mm}$$

Number of coils

$$\delta = \frac{8WD^3 n}{Gd^4}$$

or

$$60 = \frac{8 \times 11\,310 \times 200^3 \times n}{84\,000 \times 43.5^4}; \qquad n = 24.9 \approx 25 \text{ turns}$$

Example 11.4 ‖ A close-coiled helical spring absorbs 72 N·m of energy when compressed through 60 mm. There are 8 coils in the spring. The coil diameter is 10 times the wire diameter. Find the diameters of the coil and wire and the maximum shear stress. G = 82 GPa.

Solution

Given A close-coiled helical spring

$U = 72$ N·m $\qquad \delta = 60$ mm

$D = 10\,d \qquad n = 8$

$G = 82$ GPa

To find

— Wire and coil diameters

— Maximum shear stress

Determination of load

$$U = \frac{1}{2} W\delta$$

or

$$72\,000 = \frac{1}{2} W \times 60 \quad \text{or} \quad W = 2400 \text{ N}$$

Wire and coil diameters

$$\delta = \frac{8WD^3 n}{Gd^4}$$

or

$$60 = \frac{8 \times 2400 \times (10d)^3 \times 8}{82\,000 d^4} \quad \text{or} \quad d = 31.2 \text{ mm}$$

and

$$D = 312 \text{ mm}$$

Maximum shear stress

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 2400 \times 312}{\pi \times 31.2^3} = 62.8 \text{ MPa}$$

Example 11.5 ‖ A wagon weighing 35 kN moves at a speed of 3.6 km/h. Find the number of springs required in a buffer stop to absorb the energy of motion during a compression of 180 mm? The mean diameter of coils is 220 mm and the diameter of the steel rod of the spring is 24 mm. Each spring consists of 30 coils. G = 90 GPa.

Solution

Given A close-coiled helical spring

$v = 3.6$ km/h $\quad m = 35$ kN

$\delta = 180$ mm $\quad D = 220$ mm

$n = 30$ $\quad d = 24$ mm

$G = 90$ GPa

To find Number of springs

Determination of load

$$v = \frac{3.6 \times 1000}{3600} = 1 \text{ m/s}$$

$$\text{Kinetic energy } = \frac{1}{2}mv^2 = \frac{1}{2} \times \frac{35\,000}{9.81} \times 1^2$$

$$= 1784 \text{ N}\cdot\text{m or } 1784\,000 \text{ N}\cdot\text{mm}$$

Now,
$$\delta = \frac{8WD^3n}{Gd^4}$$

or
$$180 = \frac{8W \times 220^3 \times 30}{90\,000 \times (24)^4} \quad \text{or} \quad W = 2103 \text{ N}$$

Number of springs

Energy stored by one spring $= \frac{1}{2}W\delta = \frac{1}{2} \times 2103 \times 180 = 189\,270 \text{ N}\cdot\text{mm}$

Number of springs required $= \frac{1\,784\,000}{189\,270} = 9.43$, i.e., 10 (say)

Example 11.6 A weight of 240 N is dropped on to a close-coiled helical spring made of 18 mm steel wire. The spring consists of 22 coils wound to a diameter of 180 mm. If the instantaneous compression is 120 mm, determine the height of drop of the weight and the instantaneous stress produced. G = 88 GPa.

Solution

Given A close-coiled helical spring

$P = 240$ N $\quad d = 18$ mm

$n = 22$ $\quad D = 180$ mm

$\delta = 120$ mm $\quad G = 88$ GPa

To find

— Height of weight drop

— Stress

Determination of gradual load

Let W be the equivalent gradually applied load in N

$$\delta = \frac{8WD^3n}{Gd^4} \quad \text{or} \quad 120 = \frac{8W \times 180^3 \times 22}{88\,000 \times (18)^4} \quad \text{or} \quad W = 1080 \text{ N}$$

Height of drop

Now, Energy loss due to weight = Work done on the spring

$$P(h + \delta) = \frac{1}{2}\cdot W\delta \quad \text{or} \quad 240(h + 120) = \frac{1}{2} \times 1080 \times 120 \quad \text{or} \quad h = 150 \text{ mm}$$

Stress

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 1080 \times 180}{\pi \times 18^3} = 84.9 \text{ MPa}$$

Example 11.7 A close-coiled helical spring has a maximum load of 40 N and maximum shear stress induced is 100 MPa. The spring constant is 600 N/m in compression. If the solid length (coils touching) of the spring is 60 mm, determine the wire diameter, number of coils and the coil diameter. G = 32 GPa.

Solution

Given A close-coiled helical spring

$W = 40$ N $\quad \tau = 100$ MPa

$nd = 60$ mm $\quad G = 32$ GPa

s = 600 N/m = 0.6 N/mm

To find

— Wire and coil diameters
— number of coils

Relation between d and D

Relation between d and D can be obtained from the relation, $\tau = \frac{8WD}{\pi d^3}$

i.e., $$100 = \frac{8 \times 40 \times D}{\pi \times d^3} \quad \text{or} \quad D = 0.982\, d^3$$

Determination of diameters

As $$\delta = \frac{8WD^3 n}{Gd^4}$$

$\therefore$ Stiffness, $$s = \frac{W}{\delta} = \frac{Gd^4}{8D^3 n}$$

or $$0.6 = \frac{32\,000 \times d^4}{8(0.982 d^3)^3 \times 60/d}$$

$d^4 = 117.3 \quad$ or $\quad d = 3.29$ mm

$\therefore$ $D = 0.982 \times 3.29^3 = 35$ mm

Number of coils

Number of coils, $n = \frac{60}{3.29} = 18.24$

Example 11.8 A close-coiled helical spring is built-in at both ends. A force of 800 N is applied at an intermediate point such that the number of coils on one side is 10 and on the other 6. Determine the distribution of the force on two sides.

Solution

Given A close-coiled helical spring as shown in Fig. 11.3.

$W = 800$ N $\quad n_2/n_1 = 6/10 = 0.6$

To find Distribution of force on two sides

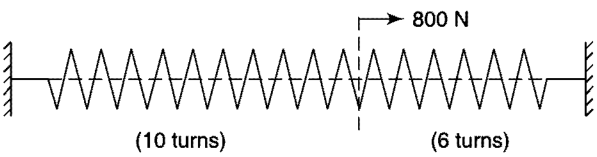

Fig.11.3

Determination of ratio of forces

As the deflection of both sides will be equal,

$$\delta = \frac{8W_1D^3n_1}{Gd^4} = \frac{8W_2D^3n_2}{Gd^4} \quad \text{or} \quad W_1n_1 = W_2n_2 \quad \text{or} \quad \frac{W_1}{W_2} = \frac{n_2}{n_1}$$

i.e. the force on each side will be in the inverse ratio of number of coils.

or $\frac{W_1}{W_2} = 0.6 \quad \text{or} \quad W_1 = 0.6W_2$

Distribution of forces

$$W_1 + W_2 = W = 800$$

$\therefore$ $0.6\,W_2 + W_2 = 800 \quad \text{or} \quad W_2 = 500 \text{ N} \quad \text{and} \quad W_1 = 300 \text{ N}$

Example 11.9 A close-coiled helical spring made of a 10-mm diameter steel bar has 8 coils of 150-mm mean diameter. Calculate the elongation, torsional stress and the strain energy per unit volume when the spring is subjected to an axial load of 130 N. G = 80 GPa.

If instead of the axial load, an axial torque of 9 N · m is applied, find the axial twist, bending stress and the strain energy per unit volume. E = 205 GPa

Solution

Given A close-coiled helical spring

W = 130 N $\qquad$ d =10 mm

n = 8 $\qquad$ D = 150 mm

T = 9 N · m $\qquad$ G = 80 GPa

E = 205 GPa

To find

— Elongation, torsional stress and strain energy under axial load

— Axial twist, bending stress strain energy under axial torque

Under axial load

$$\delta = \frac{8WD^3n}{Gd^4} = \frac{8 \times 130 \times 150^3 \times 8}{80\,000 \times 10^4} = 35.1 \text{ mm}$$

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 130 \times 150}{\pi \times 10^3} = 49.66 \text{ MPa}$$

Strain energy/unit volume $= \frac{\tau^2}{4G}$ ...(Eq. 10.5)

$$= \frac{49.66^2}{4 \times 80\,000} = 0.007\,71 \text{ N} \cdot \text{mm/mm}^3$$

or $7710 \text{ N} \cdot \text{m/m}^3 \quad \text{or} \quad 7.71 \text{ kN} \cdot \text{m/m}^3$

Under axial torque

$$\varphi = \frac{64TDn}{Ed^4} = \frac{64 \times 9000 \times 150 \times 8}{205\,000 \times 10^4} = 0.3372 \text{ rad}$$

Bending stress, $\sigma_b = \frac{32T}{\pi d^3} = \frac{32 \times 9000}{\pi \times 10^3} = 91.7 \text{ MPa}$

$$\text{Strain energy/unit volume} = \frac{T\varphi/2}{\text{Volume}}$$

$$= \frac{9000 \times 0.3372/2}{(\pi \times 10^2/4)\,\pi \times 150 \times 8} = 0.005\,124\ \text{mm/mm}^3 \text{ or } 5.124\ \text{N}\cdot\text{m/m}^3$$

Example 11.10 Show that for a close-coiled spring of circular section of mean coil diameter *D*, if *W* is the axial load applied per unit deflection and *T* the torque applied per unit angular rotation independent of *W*, then $T/W = D^2(1 + \nu)/4$ where ν is the Poisson's ratio.

Also determine the Poisson's ratio if a load of 120 N extends the spring by 80 mm and a torque of 500 N·mm produces an angular rotation of 90°. The mean coil diameter of the spring is 25 mm.

Solution

Given A close-coiled helical spring

$W = 120$ N $\qquad \delta = 80$ mm

$T = 500$ N $\qquad D = 25$ mm

$\varphi = 90°$

To find

— $T/W = D^2(1 + \nu)/4$

— Poisson's ratio

Ratio of T/D

As W is the axial load applied per unit deflection, $\therefore\ 1 = \dfrac{8WD^3n}{Gd^4}$

As T is the torque applied per unit angular rotation independent of W,

$$\therefore \qquad 1 = \frac{64TDn}{Ed^4}$$

Thus

$$\frac{8WD^3n}{Gd^4} = \frac{64TDn}{Ed^4}$$

or

$$\frac{T}{W} = \frac{D^2E}{8G} = \frac{D^2 \times 2G(1+\nu)}{8G} = \frac{D^2(1+\nu)}{4} \qquad \text{(i)}$$

Numerical

$$\text{Torque/unit angular rotation, } T = \frac{500}{90 \times (\pi/180)} = 318.3\ \text{N}\cdot\text{mm/rad}$$

$$\text{Axial load applied per unit deflection, } W = \frac{120}{80} = 1.5\ \text{N}$$

Thus from (i),

$$\frac{318.3}{1.5} = \frac{25^2(1+\nu)}{4}$$

or

$$1 + \nu = \frac{848.8}{25^2} = 1.358 \quad \text{or} \quad \nu = 0.358$$

Example 11.11 A close-coiled helical spring has its free length as 120 mm. It absorbs 40 N·m of energy when fully compressed and the coils are in contact. The mean coil diameter is 80 mm. Determine the diameter of the steel wire required and the number of coils if the maximum shear stress is to be 120 MPa. G = 82 GPa.

Solution

Given A close-coiled helical spring

$U = 40\ \text{N}\cdot\text{m}$ $\quad D = 80$ mm

$l = 120$ N $\quad \tau = 120$ MPa

$G = 82$ GPa

To find

— Wire diameter

— number of coils

Relation for deflection

Deflection = Free length – solid length = 120 – *nd*,

We have $\delta = \dfrac{8WD^3n}{Gd^4}$ or $120 - nd = \dfrac{8W \times 80^3 \times n}{82\,000 \times d^4}$

or $(120 - nd)d^4 = 49.95Wn$ (i)

Relation for shear stress

W and n can be expressed in terms of d as follows:

$$\tau = \frac{8WD}{\pi d^3} \quad \text{or} \quad W = \frac{\pi\tau d^3}{8D} = \frac{\pi \times 120 \times d^3}{8 \times 80} = 0.589d^3$$

Relation for strain energy

$$U = \frac{\tau^2}{4G} \times \text{Volume} = \frac{\tau^2}{4G} \times \text{Area of cross-section} \times \text{length}$$

$$40\,000 = \frac{120^2}{4 \times 82\,000} \times \frac{\pi}{4} \times d^2 \times \pi \times 80 \times n \quad \text{or} \quad n = \frac{4616}{d^2}$$

Determination of d and n

From (i), $\left(120 - \dfrac{4616}{d^2}d\right)d^4 = 49.95 \times 0.589d^3 \times \dfrac{4616}{d^2}$

or $120d^3 - 4616d^2 = 135\,805$ or $d^3 - 38.47d^2 = 1132$

Solving by trial and error, $d = 39.2$ m and $n = \dfrac{4616}{39.2^2} = 3$

11.2 SPRINGS IN SERIES AND PARALLEL

- When two springs of different stiffnesses are joined end to end, they are said to be connected in *series*. For springs in series, the load is the same for both the springs whereas the deflection is the sum of deflection of each,

  i.e., $\delta = \delta_1 + \delta_2$ or $\dfrac{W}{s} = \dfrac{W}{s_1} + \dfrac{W}{s_2}$ or $\dfrac{1}{s} = \dfrac{1}{s_1} + \dfrac{1}{s_2}$ or $s = \dfrac{s_1 s_2}{s_1 + s_2}$

  where s_1 and s_2 are the stiffness of each joined spring and s is the equivalent stiffness of the composite spring.
- When two springs of different stiffnesses are joined in *parallel*, they have the common deflection and the load is the sum of load taken by each,

  i.e., common deflection $\delta = \dfrac{W}{s} = \dfrac{W_1}{s_1} = \dfrac{W_2}{s_2}$

and $W = W_1 + W_2$ or $s\delta = s_1\delta + s_2\delta$ or $s = s_1 + s_2$

where W_1 and W_2 are the loads taken by each spring.

Example 11.12 A composite spring consists of two close-coiled helical springs connected in series. Each spring has 14 coils at a mean diameter of 20 mm. The stiffness of the composite spring is 800 N/m. If the wire diameter of one spring is 2.5 mm, find the wire diameter of the other.

What will be the maximum load which can be carried by the composite spring and the corresponding deflection for a maximum shear stress of 150 MPa? G = 78 GPa.

Solution

Given A composite spring having two close-coiled helical springs in series

$n = 14$ $\quad D = 20$ mm

$d_1 = 2.5$ mm $\quad \tau = 150$ MPa

$G = 78$ GPa $\quad s = 800$ N/m = 0.8 N/mm

To find

— Wire diameter d_2

— Maximum load and deflection

Determination of wire diameter

For springs in series, $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$

$$s_1 = \frac{W}{\delta} = \frac{Gd_1^4}{8D^3n} = \frac{78\,000 \times 2.5^4}{8X20^3 \times 14} = 3.4$$

and $$s_2 = \frac{78\,000 \times d_2^4}{8 \times 20^3 \times 14} = 0.087\,05\, d_2^4$$

$\therefore$ $$\frac{1}{0.8} = \frac{1}{3.4} + \frac{1}{0.087\,05\, d_2^4}$$

or $$\frac{1}{0.087\,05\, d_2^4} = \frac{1}{0.8} - \frac{1}{3.4} = 0.956$$

or $d_2^4 = 12.02$ or $d_2 = 1.86$ mm

Load and deflection

$$\tau = \frac{8WD}{\pi d_2^3} \quad \text{or} \quad 150 = \frac{8W \times 20}{\pi \times 1.86^3} \quad \text{or} \quad W = 18.95 \text{ N}$$

Total deflection $= \frac{W}{s} = \frac{18.95}{0.8} = 23.7$ mm

Example 11.13 A load W is transmitted to a combination of three springs through a weightless bar as shown in Fig. 11.4. All the springs are made from the same bar of steel and are of the same length initially. The number of coils in the three springs are 20, 24 and 30 and the mean diameters are in the ratio 5:6:7 respectively. Determine the value of x so that the bar remains horizontal after applying the load.

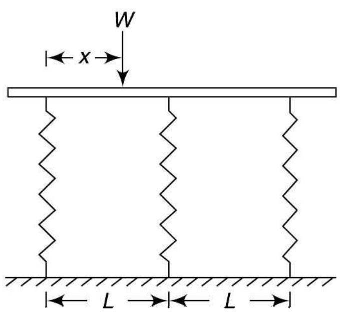

Fig. 11.4

Solution

Given A composite spring having two close-coiled helical springs in series as shown in Fig. 11.4.

$n_1 = 20 \qquad n_2 = 24 \qquad n_3 = 30 \qquad D_1 : D_2 : D_3 = 5:6:7 = 1:1.2:1.4$

To find Value of x

The bar remains horizontal after applying the load implying that the deflection is equal for all the three springs. Let W_1, W_2 and W_3 be the loads shared by the three springs.

Sharing of load by wires

We have $\delta = \dfrac{8W_1 D_1^3 n_1}{Gd^4} = \dfrac{8W_2 D_2^3 n_2}{Gd^4} = \dfrac{8W_3 D_3^3 n_3}{Gd^4}$

From first and second terms, $W_2 = W_1 \left(\dfrac{D_1}{D_2}\right)^3 \dfrac{n_1}{n_2} = W_1 \left(\dfrac{1}{1.2}\right)^3 \times \dfrac{20}{24} = 0.4823 W_1$

From first and third terms, $W_3 = W_1 \left(\dfrac{D_1}{D_3}\right)^3 \dfrac{n_1}{n_3} = W_1 \left(\dfrac{1}{1.4}\right)^3 \times \dfrac{20}{30} = 0.243 W_1$

Determination of x

$$W = W_1 + W_2 + W_3 = W_1 + 0.4823 W_1 + 0.243 W_1 = 1.7253 W_1$$

Taking moments about the line of force of the first load,

$$W \cdot x = W_2 L + W_3 \times 2L$$

or $\quad 1.7253 W_1 \cdot x = 0.4823 W_1 L + 0.243 W_1 \times 2L \quad$ or $\quad x = 0.5612\, L$

11.3 CONCENTRIC (CLUSTER) SPRINGS

Concentric or cluster springs are those which are close-coiled helical springs and one placed inside the other as shown in Fig. 11.5.

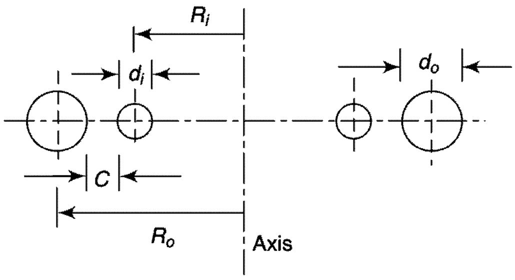

Fig. 11.5

Let $\quad W_i, W_o$ = Load shared by the inner and outer springs respectively
D_i, D_o = Mean coil diameters of the inner and outer springs respectively
d_i, d_o = Wire diameters of the inner and outer springs respectively
n_i, n_o = Number of coils in the outer and inner springs respectively
c = Clearance between the inner and the outer springs

From Fig. 11.5, $\quad R_o = R_i + \dfrac{d_i}{2} + c + \dfrac{d_o}{2} \quad$ or $\quad D_o = D_i + d_i + 2c + d_o$

- If initially the free length of the springs is the same, the deflection will be the same.

and $$\delta = \frac{8W_i D_i^3 n_i}{G_i d_i^4} = \frac{8W_o D_o^3 n_o}{G_o d_o^4} \qquad \text{(i)}$$

- If the maximum shear stress is equal in the two springs,

$$\tau = \frac{8W_i D_i}{\pi d_i^3} = \frac{8W_o D_o}{\pi d_o^3} \quad \text{(ii)}$$

Dividing (ii) by (i), $\frac{G_i d_i}{n_i D_i^2} = \frac{G_o d_o}{n_o D_o^2}$ or $\left(\frac{D_o}{D_i}\right)^2 = \frac{n_i}{n_o}\frac{G_o}{G_i}\frac{d_o}{d_i}$

If the material and the solid length is the same i.e. $G_i = G_o$ and $n_i d_i = n_o d_o$

$$\therefore \quad \left(\frac{D_o}{D_i}\right)^2 = \left(\frac{d_o}{d_i}\right)^2 \quad \text{or} \quad \frac{D_o}{D_i} = \frac{d_o}{d_i} \quad \text{(iii)}$$

From (ii), $\frac{W_o}{W_i} = \frac{D_i}{D_o}\frac{d_o^{\;3}}{d_i^{\;3}} = \frac{d_i}{d_o}\left(\frac{d_o}{d_i}\right)^3 = \left(\frac{d_o}{d_i}\right)^2$

Also using (iii), $\frac{W_o}{W_i} = \left(\frac{D_o}{D_i}\right)^2$

Example 11.14 In a compound helical spring, the inner ring is concentric with the outer one. The maximum permissible deflection of the springs is 30 mm and the solid length is 40 mm. The springs are subjected to an axial load of 5 kN. Both the springs are made of the same material having G = 82 GPa. The diameter of the inner spring is 120 mm and the radial clearance is 2 mm. What will be the wire diameters and the load shared by each spring if the allowable shear stress is 180 MPa?

Solution

Given A compound helical spring

$\delta = 30$ mm	$n_i d_i = n_o d_o = 40$ mm
$W = 5$ kN	$D_i = 120$ mm
$c = 2$ mm	$\tau = 180$ MPa
$G = 82$ GPa	

To find

— Wire diameters

— Load shared by each spring

Determination of wire diameters

- When the springs are fully compressed, both the springs attain the maximum value of stress.

Then $$\tau = \frac{8W_i D_i}{\pi d_i^3} = \frac{8W_o D_o}{\pi d_o^3} = 180 \quad \text{(i)}$$

- Deflection is same in both springs, $\delta = \frac{8W_i D_i^3 n_i}{G d_i^4} = \frac{8W_o D_o^3 n_o}{G d_o^4} = 30$ (ii)

Dividing (i) by (ii), $\frac{G d_i}{n_i \pi D_i^2} = \frac{180}{30} = 6$

or $\frac{82\,000 d_i}{(40/d_i)\pi D_i^2} = 6$ ..... (solid length is same, $n_i d_i = n_o d_o = 40$)

or $\left(\frac{D_i}{d_i}\right)^2 = 108.76$ or $\frac{D_i}{d_i} = 10.43$ (iii) or $$d_i = \frac{120}{10.43} = 11.5 \text{ mm}$$

In a similar way, it can be shown that $\frac{D_o}{d_o} = 10.43$ (iv)

$$D_o = D_i + d_i + 2c + d_o$$

or $10.43d_o = 120 + 11.5 + 2 \times 2 + d_o$ ...(Refer Fig. 11.6)

or $d_o = 14.37$ mm

and $D_o = 150$ mm

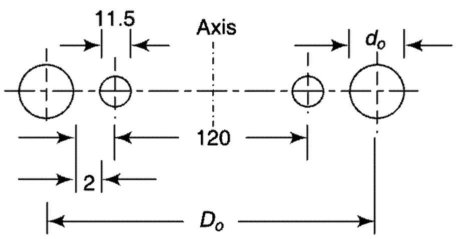

Fig. 11.6

Load sharing

From (iii) and (iv), $\frac{D_o}{d_o} = \frac{D_i}{d_i}$ or $\frac{D_o}{D_i} = \frac{d_o}{d_i}$

From (i), $\frac{W_i}{W_o} = \frac{D_o}{D_o} \cdot \frac{d_i^3}{d_o^3} = \frac{d_i^2}{d_o^2} = \left(\frac{11.5}{14.37}\right)^2 = 0.64$

Also, $W_i + W_o = 5$ or $0.65W_o + W_o = 5$

or $W_o = 3.05$ kN and $W_i = 1.95$ kN

Example 11.15 A compound spring consists of two co-axial close-coiled springs with the same initial length. The outer spring has 20 turns of 15-mm diameter wire coiled to a mean diameter of 140 mm and the inner spring has 28 turns with a mean diameter of 100 mm. Determine the diameter of the wire for the inner spring and the stiffness of the compound spring if the working stress in each spring is the same. G = 84 GPa.

Solution

Given A compound helical spring

$n_o = 20$	d_o =15 mm
D_o =140 mm	$n_i = 28$
D_i =100 mm	G =84 GPa

To find

— Wire diameter of inner spring

— Stiffness of compound spring

Determination of inner diameter

- The working stress in each spring is the same,

$$\therefore \quad \frac{8W_iD_i}{\pi d_i^3} = \frac{8W_oD_o}{\pi d_o^3} \quad \text{(i)}$$

- As the initial length of both the springs is the same, deflection is to be the same in both springs,

$$\frac{8W_iD_i^3n_i}{Gd_i^4} = \frac{8W_oD_o^3n_o}{Gd_o^4} \quad \text{(ii)}$$

Dividing (i) by (ii), $\frac{d_i}{n_iD_i^2} = \frac{d_o}{n_oD_o^2}$ or $n_iD_i^2d_o = n_oD_o^2d_i$

or $28 \times 100^2 \times 15 = 20 \times 140^2 \times d_i$ or $d_i = 10.7$ mm

Determination of stiffness

Stiffness of inner spring, $\frac{W_i}{\delta_i} = \frac{W_i}{8W_iD_i^3n_i/Gd_i^4}$

$$= \frac{Gd_i^4}{8D_i^3n_i} = \frac{84\,000 \times 10.7^4}{8 \times 100^3 \times 28} = 4.92 \text{ N/mm}$$

Stiffness of outer spring = $\frac{W_o}{\delta_o} = \frac{Gd_o^4}{8D_o^3n_o} = \frac{84\,000 \times 15^4}{8 \times 140^3 \times 20} = 9.69$ N/mm

As the springs are in parallel, the combined stiffness = 4.92 + 9.69 = 14.61 N/mm

Example 11.16 Two close-coiled concentric helical springs of equal length are made out of the same wire of 12-mm diameter and support a compressive load of 1.2 kN. The outer spring consists of 20 turns of 240-mm mean diameter and the inner spring has 24 turns of 200-mm mean diameter. Determine the maximum stress produced in each spring.

Solution

Given A compound helical spring

$d_i = d_o = 12$ mm $\qquad W = 1.2$ kN

$n_o = 20 \qquad D_o = 240$ mm

$n_i = 24 \qquad D_i = 200$ mm

To find Maximum shear stress in each spring

Determination of load distribution

Deflection is to be the same in both springs,

$\therefore \qquad \delta = \frac{8W_iD_i^3n_i}{G_id_i^4} = \frac{8W_oD_o^3n_o}{G_od_o^4}$

or $\qquad W_iD_i^3n_i = W_oD_o^3n_o \qquad$ (For same material, $G_i = G_o$ and $d_i = d_o = 12$ mm)

$W_i \times 200^3 \times 24 = W_o \times 240^3 \times 20$

or $\qquad W_i = 1.44W_o$

But $\qquad W_i + W_o = 1200 \qquad$ or $\qquad 1.4\,W_o + W_o = 1200$

or $\qquad W_o = 491.8$ N

and $\qquad W_i = 1200 - 491.8 = 708.2$ N

Maximum shear stresses

Maximum shear stress in outer spring = $\frac{8W_oD_o}{\pi d_o^3} = \frac{8 \times 491.8 \times 240}{\pi \times 12^3} = 174$ MPa

Maximum shear stress in inner spring = $\frac{8W_iD_i}{\pi d_i^3} = \frac{8 \times 708.8 \times 200}{\pi \times 12^3} = 209$ MPa

Example 11.17 The inner spring of a compound helical spring is concentric with the outer one but is 8 mm shorter than that. The outer spring is of 30-mm mean diameter, and has 12 coils with a wire diameter of 4 mm. If an axial load of 250 N compresses the outer spring by 20 mm, determine the stiffness of the inner spring.

What will be the wire diameter of the inner spring if it has 10 coils and the radial clearance between the springs is 2 mm? G = 78 GPa.

Solution

Given A compound helical spring

$\delta_o - \delta_i = 8$ mm $D_o = 30$ mm
$n_o = 12$ $d_o = 4$ mm
$W = 250$ N $\delta_o = 20$ mm
$n_i = 10$ $c = 2$ mm
$G = 78$ GPa

To find

— Stiffness of inner spring
— Wire diameter of inner spring

Stiffness of inner spring

For outer spring, $$\delta_o = \frac{8WD_o^3 n}{Gd_o^4}$$

or $$20 = \frac{8 \times W \times 30^3 \times 12}{78\,000 \times 4^4} \quad \text{or} \quad W = 154.1 \text{ N}$$

Load carried by the inner spring = 250 – 154.1 = 95.9 N
Compression of the outer spring = 20 mm
∴ compression of the inner spring = 20 – 8 = 12 mm

Thus stiffness of the inner spring, $\frac{95.9}{12} = 8$ N/mm

Wire diameter of inner spring

$$D_o = D_i + d_i + 2c + d_o \quad \text{or} \quad 30 = D_i + d_i + 2 \times 2 + 4 \qquad \text{...(Refer Fig. 11.7)}$$

or $$D_i = 22 - d_i$$

For inner spring,

$$\delta = \frac{8WD_i^3 n}{Gd_i^4} \quad \text{or} \quad 12 = \frac{8 \times 95.9 \times (22 - d_i)^3 \times 10}{78\,000 \times d_i^4}$$

or $$122 d_i^4 = (22 - d_i)^3 \quad \text{or} \quad d_i = [(22 - d_i)^3/122]^{0.25}$$

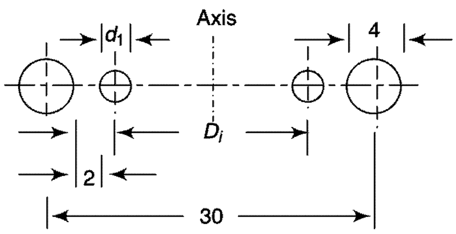

Fig. 11.7

The equation can be solved by trial and error or use the following technique:

As d_i is small compared to 22,

For first approximation, take $d_i = 0$, $d_i = [(22 - 0)^3/122]^{0.25}$ or $d_i = 3.06$

For second approximation, take $d_i = 3.06$, $d_i = [(22 - 3.06)^3/122]^{0.25}$ or $d_i = 2.73$

For third approximation, take $d_i = 2.73$, $d_i = [(22 - 2.73)^3/122]^{0.25}$ or $d_i = 2.76$

For final approximation, take $d_i = 3.76$, $d_i = [(22 - 2.76)^3/122]^{0.25}$ or $d_i = 2.763$

Thus $d_i = 2.763$ mm

11.4 OPEN-COILED HELICAL SPRINGS

Let α be the angle of helix and let the following axes be in a vertical plane tangential to the helix at O (Fig. 11.6).

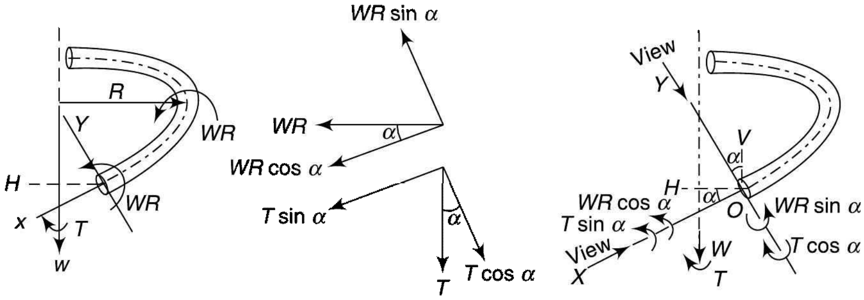

Fig. 11.8

OH = Horizontal axis
OV = Vertical axis
OX= Polar axis or axis of twist at any normal cross-section inclined at angle α to OH
OY = Bending axis about which bending takes place inclined at angle α to OV

l = Length of wire = $\dfrac{2\pi Rn}{\cos\alpha} = \dfrac{\pi Dn}{\cos\alpha}$

Apply an axial load W and an axial torque T to the spring.

The axial load W produces a couple WR about OH which can be represented by a vector by taking a vector along the axis of the couple and using the right-hand screw rule for the direction, i.e., for a clockwise couple, a vector away from the viewer. The components of the vector WR along X and Y are

- $WR\cos\alpha$ that tends to twist the wire
- $WR\sin\alpha$ that tends to open up the coil or to reduce the curvature

The axial torque T about OV has the following components along X and Y

- $T\sin\alpha$ that tends to twist the wire
- $T\cos\alpha$ that tends to close up the coil or to increase the curvature

Thus combined twisting couple, $T' = WR\cos\alpha - WR\sin\alpha$
and combined bending couple tending to increase the curvature

$$M = T\cos\alpha - WR\sin\alpha$$

Total strain energy, $U = \dfrac{(WR\cos\alpha + T\sin\alpha)^2 l}{2GJ} + \dfrac{(T\cos\alpha - WR\sin\alpha)^2 l}{2EI}$ (11.7)

Axial deflection and axial rotation may be obtained using Castigliano's theorem.

Axial Deflection

The axial deflection, $\delta = \dfrac{\partial U}{\partial W}$

$$= \frac{2l(WR\cos\alpha + T\sin\alpha)R\cos\alpha}{2GJ} + \frac{2l(T\cos\alpha - WR\sin\alpha)(-R\sin\alpha)}{2EI}$$

$$= WR^2 l\left(\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right) + TRl\left(\frac{1}{GJ} - \frac{1}{EI}\right)\sin\alpha\cos\alpha$$

(i) For axial load only ($T = 0$),

$$\delta = WR^2 l\left(\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{GJ}\right)$$

$$= W \cdot \frac{D^2}{4} \cdot \frac{\pi Dn}{\cos\alpha}\left(\frac{\cos^2\alpha}{G(\pi d^4/32)} + \frac{\sin^2\alpha}{E(\pi d^4/64)}\right)$$

$$= W \cdot \frac{D^2}{4} \cdot \frac{\pi Dn}{\cos\alpha} \cdot \frac{32}{\pi d^4}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{8WD^3 n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

(ii) For axial torque only ($W = 0$),

$$\delta = TRl\left(\frac{1}{GJ} - \frac{1}{EI}\right)\sin\alpha\cos\alpha$$

$$= T \cdot \frac{D}{2} \cdot \frac{\pi Dn}{\cos\alpha}\left(\frac{1}{G(\pi d^4/32)} + \frac{1}{E(\pi d^4/64)}\right)\sin\alpha\cos\alpha$$

$$= T \cdot \frac{D}{2} \cdot \frac{\pi Dn}{\cos\alpha} \cdot \frac{32}{\pi d^4}\left(\frac{1}{G} - \frac{2}{E}\right)\sin\alpha\cos\alpha$$

$$= \frac{16TD^3 n\sin\alpha}{d^4 \cdot \cos\alpha}\left(\frac{1}{G} - \frac{2}{E}\right)$$

Axial Rotation

The axial rotation, $\varphi = \dfrac{\partial U}{\partial T}$

$$= \frac{2l(WR\cos\alpha + T\sin\alpha)\sin\alpha}{2GJ} + \frac{2l(T\cos\alpha - WR\sin\alpha)\cos\alpha}{2EI}$$

$$= WRl\sin\alpha\cos\alpha\left(\frac{1}{GJ} - \frac{1}{EI}\right) + Tl\left(\frac{\sin^2\alpha}{GJ} - \frac{\cos^2\alpha}{EI}\right)$$

(i) For axial load only ($T = 0$),

$$\varphi = WRl\sin\alpha\cos\alpha\left(\frac{1}{GJ} - \frac{1}{EI}\right)$$

$$= W\frac{D}{2}\frac{\pi Dn}{\cos\alpha} \cdot \sin\alpha\cos\alpha\left(\frac{1}{G(\pi d^4/32)} - \frac{1}{E(\pi d^4/64)}\right)$$

$$= W\frac{D}{2} \cdot \frac{\pi Dn}{\cos\alpha} \cdot \frac{32}{\pi d^4} \cdot \sin\alpha\cos\alpha\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16WD^2 n \cdot \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right) \qquad (11.8)$$

(ii) For axial torque only ($W = 0$),

$$\varphi = Tl\left(\frac{\sin^2\alpha}{GJ} - \frac{\cos^2\alpha}{EI}\right)$$

$$= T \cdot \frac{\pi D n}{\cos \alpha}\left(\frac{\sin^2 \alpha}{G(\pi d^4/32)} - \frac{\cos^2 \alpha}{E(\pi d^4/64)}\right)$$

$$= T \cdot \frac{\pi D n}{\cos \alpha} \cdot \frac{32}{\pi d^4}\left(\frac{\sin^2 \alpha}{G} - \frac{2\cos^2 \alpha}{E}\right)$$

$$= \frac{32 T D n}{d^4 \cdot \cos \alpha}\left(\frac{\sin^2 \alpha}{G} + \frac{2\cos^2 \alpha}{E}\right) \quad (11.9)$$

Stresses

Combined twisting couple, $T' = WR \cos \alpha + T \sin \alpha$
Combined bending couple, $M = \text{T} \cos \alpha - WR \sin \alpha$

$$\tau = \frac{16T'}{\pi d^3} \quad \text{and} \quad \sigma_b = \frac{32M}{\pi d^3}$$

(i) Axial load only, $T' = WR \cos \alpha$ or $\tau = \dfrac{16WR \cos \alpha}{\pi d^3}$

and $M = WR \sin \alpha$ or $\sigma_b = \dfrac{32WR \sin \alpha}{\pi d^3}$

(ii) Axial torque only, $T' = T \sin \alpha$ or $\tau = \dfrac{16T \sin \alpha}{\pi d^3}$

and $M = T \cos \alpha$ or $\sigma_b = \dfrac{32T \cos \alpha}{\pi d^3}$

$$\text{Principal stresses} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2}$$

$$\text{Maximum shear stress} = \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} \quad (11.10)$$

Example 11.18 An open-coiled helical spring has 12 turns wound to a mean diameter of 100 mm. The angle of the coils with a plane perpendicular to the axis of the coil is 30°. The wire diameter is 8 mm. Determine

(i) the axial extension with a load of 80 N
(ii) the angle turned by the free end if free to rotate

E = 205 MPa and G = 80 GPa.

Solution

Given A open-coiled helical spring

D = 100 mm $\quad n = 12$
d = 8 mm $\quad \alpha = 30°$
W = 80 N $\quad G$ = 80 GPa
E = 205 MPa

To find

— Axial extension

— Angle turned by free end

Axial extension

$$\delta = \frac{8WD^3n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{8 \times 80 \times 100^3 \times 12}{8^4 \cdot \cos 30^o}\left(\frac{\cos^2 30^\circ}{80\,000} + \frac{2\sin^2 30^\circ}{205\,000}\right)$$

$$= 2.165 \times 10^6 (9.375 \times 10^{-6} + 2.439 \times 10^{-6}) = 25.88 \text{ mm}$$

Angle turned

$$\varphi = \frac{16WD^2n\sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16 \times 80 \times 100^2 \times 12 \sin 30^\circ}{8^4}\left(\frac{1}{80\,000} - \frac{2}{205\,000}\right)$$

$$= 18\,750 \times 2.744 \times 10^{-6} = 0.0515 \text{ rad} = \left(0.0515 \times \frac{180}{\pi}\right)^\circ = 2.95^\circ$$

Example 11.19 An open-coiled helical spring has 10 coils and is made out of a 12 mm diameter steel rod. The mean diameter of the coils is 80 mm and the helix angle 15°. Find the deflection under an axial load of 250 N. What are the maximum intensities of direct and shear stresses induced in the section of the wire?

If the above axial load is replaced by an axial torque of 60 N · m, determine the axial deflection and the angle of rotation about the axis of the coil. G = 80 MPa and E = 204 GPa.

Solution

Given A open-coiled helical spring

$D = 80$ mm $\qquad n = 10$

$d = 12$ mm $\qquad \alpha = 15^\circ$

$W = 250$ N

~~~~~~

$T = 60$ N · m $\qquad G = 80$ GPa $\qquad E = 204$ MPa

To find

— Deflection and direct and shear stresses

— Deflection and angle of rotation

For axial load

(i) $$\delta = \frac{8WD^3n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} - \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{8 \times 250 \times 80^3 \times 10}{12^4 \cdot \cos 15^\circ}\left(\frac{\cos^2 15^\circ}{80\,000} + \frac{2\sin^2 15^\circ}{204\,000}\right) = 6.3 \text{ mm}$$

(ii) $T = WR\cos\alpha = 250 \times 40 \times \cos 15^\circ = 9659$ N · mm

$M = WR\sin\alpha = 250 \times 40 \times \sin 15^\circ = 2588$ N · mm
~~~~~~

$$\tau = \frac{16T}{\pi d^3} = \frac{16 \times 9659}{\pi \times 12^3} = 28.47 \text{ MPa}$$

$$\sigma_b = \frac{32M}{\pi d^3} = \frac{32 \times 2588}{\pi \times 12^3} = 15.26 \text{ MPa}$$

$$\text{Principal stresses} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} = \frac{15.26}{2} \pm \sqrt{\left(\frac{15.26}{2}\right)^2 + 28.47^2} = 7.63 \pm 29.47$$

$$= 37.1 \text{ N/mm}^2 \text{ and } -21.84 \text{ MPa}$$

$$\text{Maximum shear stress} = \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{15.26}{2}\right)^2 + 28.47^2} = 29.47 \text{ MPa}$$

For axial torque

If axial load is replaced by an axial torque of 60 N · m,

$$\delta = \frac{16TD^2 n \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16 \times 60\,000 \times 80^2 \times 10 \sin 15°}{12^4}\left(\frac{1}{80\,000} - \frac{2}{204\,000}\right) = 2.07 \text{ mm}$$

$$\varphi = \frac{32TDn}{d^4 \cdot \cos\alpha}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right)$$

$$= \frac{32 \times 60\,000 \times 80 \times 10}{12^4 \cdot \cos 15°}\left(\frac{\sin^2 15°}{80\,000} + \frac{2\cos^2 15°}{204\,000}\right)$$

$$= 0.765 \text{ rad or } \left(0.765 \times \frac{180}{\pi}\right)^{\circ} = 43.9°$$

Example 11.20 In an open-coiled helical spring, the stresses due to twisting and bending are 120 MPa and 90 MPa respectively when the spring is loaded axially. The springs consists of 8 coils and the mean diameter of the coils is 10 times the diameter of the wire. Determine the maximum permissible load and the diameter of wire for a maximum deflection of 30 mm. G = 80 GPa and E = 204 GPa.

Solution

Given A open-coiled helical spring

$\tau = 120$ MPa	$\sigma_b = 90$ MPa
$n = 8$	$D = 10d$
$\delta = 30$ mm	$G = 80$ GPa
$E = 204$ MPa	

To find

— Maximum load
— Wire diameter

$$D = 10d \quad \text{or} \quad R = 5d$$

Bending and shear stress constraints

$$\tau = \frac{16WR\cos\alpha}{\pi d^3}$$

or $$120 = \frac{16W(5d)\cos\alpha}{\pi d^3} = \frac{80W\cos\alpha}{\pi d^2} \quad \text{(i)}$$

and $$\sigma_b = \frac{32WR\sin\alpha}{\pi d^3}$$

or $$90 = \frac{16W(5d)\sin\alpha}{\pi d^3} = \frac{160W\sin\alpha}{\pi d^2} \quad \text{(ii)}$$

Dividing (ii) by (i), $\tan\alpha = 0.375$ or $\alpha = 20.56°$

From (i), $120 = \dfrac{80W\cos 20.56°}{\pi d^2}$ or $W = 5.033\, d^2$

Deflection constraints

$$\delta = \frac{8WD^3 n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$30 = \frac{8 \times 5.033(10d)^3 \times 8}{d^4 \cdot \cos 20.56°}\left(\frac{\cos^2 20.56°}{80\,000} + \frac{2\sin^2 20.56°}{204\,000}\right)$$

or $$30 = 344\,024(10.958 \times 10^{-6} + 1.203 \times 10^{-6})$$

or $$d = 7.17 \text{ mm}$$

$$W = 5.033 \times 7.17^2 = 258.7 \text{ N}$$

Example 11.21 The mean coil diameter of an open-coiled helical spring is 96 mm and the pitch of the coils is 100 mm. The diameter of the wire is 12 mm and the number of coils is 10. Find the axial torque on the spring which will produce a shear stress of 210 MPa. Also determine the angular twist due to this torque. E = 204 GPa and G = 82 GPa.

Solution

Given A open-coiled helical spring

$D = 96$ mm	$p = 100$ mm
$d = 12$ mm	$n = 10$
$\tau = 210$ MPa	$G = 82$ GPa
$E = 204$ MPa	

To find

— Axial torque

— Angular twist

$$\tan\alpha = \frac{p}{2\pi R} = \frac{100}{2\pi \times 48} = 0.3316 \quad \text{or} \quad \alpha = 18.34°$$

Axial torque

Let the axial torque be T. Then

$$\tau = \frac{16T\sin\alpha}{\pi d^3} \quad \text{and} \quad \sigma_b = \frac{32T\cos\alpha}{\pi d^3}$$

$$\text{Maximum shear stress} = \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2}$$

$$= \sqrt{\left(\frac{32T\cos\alpha}{2\pi d^3}\right)^2 + \left(\frac{16T\sin\alpha}{\pi d^3}\right)^2}$$

or $$210 = \frac{16T}{\pi \times 12^3} \quad \text{or} \quad T = 71\,251\text{ N}$$

Angular twist

$$\varphi = \frac{32TDn}{d^4 \cdot \cos\alpha}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right)$$

$$= \frac{32 \times 71\,251 \times 96 \times 10}{12^4 \times \cos 18.34°}\left(\frac{\sin^2 18.34°}{82\,000} + \frac{2\cos^2 18.34°}{204\,000}\right)$$

$$= 111\,206(1.2074 \times 10^{-6} + 8.8333 \times 10^{-6})$$

$$= 1.117\text{ rad} = \left(1.117 \times \frac{180}{\pi}\right)^\circ = 64°$$

Example 11.22 In case the formula for the deflection of a close-coiled helical spring is used instead of that for an open-coiled spring, find the maximum angle of helix for which the error does not exceed by 2%, given that $E = 2G$.

Solution

Given $\frac{\delta_o - \delta_c}{\delta_o} = 0.02 \qquad E = 2G$

To find Maximum helix angle

Deflection formulae

For a closed-coiled spring, $\delta_c = \frac{8WD^3n}{Gd^4}$

and for an open-coiled spring, $\delta_o = \frac{8WD^2n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$

Maximum helix angle

Now, $$\frac{\delta_o - \delta_c}{\delta_o} = 0.02 \quad \text{or} \quad 1 - \frac{\delta_c}{\delta_o} = 0.02$$

or $$1 - \frac{\frac{8WD^2n}{Gd^4}}{\frac{8WD^2n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)} = 0.02$$

$$1 - \frac{E\cos\alpha}{E\cos^2\alpha + 2G\sin^2\alpha} = 0.02$$

or $$1 - \frac{2G\cos\alpha}{2G\cos^2\alpha + 2G\sin^2\alpha} = 0.02$$

or $\cos\alpha = 0.98$ or $\alpha = 11.48°$

FLAT SPIRAL SPRINGS

A flat spiral type of spring consists of a uniform thin strip wound into a spiral in one plane and pinned at the outer end (Fig. 11.9). It is wound by applying a torque with a spindle which turns the inner end at the centre. The wound spring is released slowly over a period of time. Such type of springs finds their use in clocks.

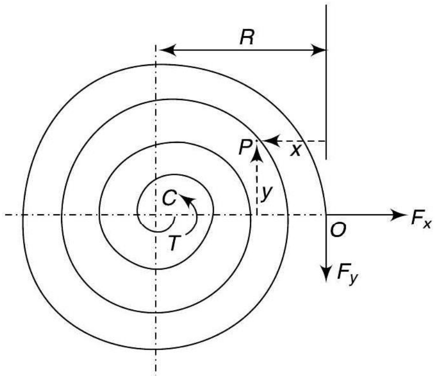

Fig. 11.9

Let T = Torque tending to wind up the spring

R = Maximum radius of the spiral

F_x = Reaction at the outer end in x-direction

F_y = Reaction at the outer end in y-direction

Taking moments about the spindle axis, $T = F_y \cdot R$

Assuming the centre to be at fixed end O,

Bending moment at any point P (x,y) on the spiral,

$$M = F_y \cdot x - F_y \cdot y$$

(tending to increase the radius)

Strain energy of the spring may be found by considering an infinitesimal length ds at P,

$$U = \int \frac{M^2}{2EI} \cdot ds = \frac{1}{2EI} \int (F_y \cdot x - F_x \cdot y)^2 \cdot ds \qquad \text{(i)}$$

As O is the fixed end of the spiral strip, deflection in x-direction is zero. Applying Castigliano's theorem,

$$\frac{\partial U}{\partial F_x} = 0$$

or $$\int \frac{(F_y \cdot x - F_x \cdot y)(-y)}{EI} \cdot ds = 0$$

or $$\int (F_y \cdot x - F_x \cdot y)(-y) ds = 0$$

or $$F_x \int y^2 \cdot ds - F_y \int xy \cdot ds = 0$$

or $$F_x = \frac{F_x \int xy \cdot ds}{\int y^2 \cdot ds} = \frac{F_x \int xy \cdot ds}{\int y^2 . ds}$$

Due to symmetry of the spiral curve about the x-axis, $F_x \int xy \cdot ds = 0$

$\therefore$ $$F_x = 0$$

Thus $$U = \frac{1}{2EI} \int (F_y \cdot x)^2 \cdot ds = \int \frac{(T \cdot x/R)^2}{2EI} \cdot ds \qquad (\text{as } T = F_y \cdot R)$$

If θ is the rotation of the spiral,

$$\theta = \frac{\partial U}{\partial T} = \frac{T}{EIR^2} \int x^2 \cdot ds$$

The integral $\int x^2 \cdot ds$ can approximately be found by assuming the spiral to be a disc.

Let l, t denote the length and thickness of the spiral. Also let A be the area of the disc.

Area of the infinitesimal length, $dA = ds \cdot t$

Total area of the disc = $l \cdot t$ (approximately).

Now, $\int x^2 \cdot ds$ = Second moment of area about vertical axis through O

= Second moment of area about y-axis through $C + AR^2$

Thus, $\int x^2 \cdot ds = \frac{1}{t}\int x^2 \cdot dA = \frac{1}{t}\left(\frac{\pi}{64}D^4 + AR^2\right) = \frac{1}{t}\left(A \cdot \frac{R^2}{4} + AR^2\right)$

$$= \frac{1.25l \cdot t \cdot R^2}{t} = 1.25R^2 t$$

Therefore, $$\theta = \frac{1.25R^2 lT}{EIR^2} = \frac{1.25Tl}{EI} \quad (11.11)$$

Maximum bending moment = $F_y \cdot 2R = 2T$ at the left-hand edge of the spiral

If b is the width and t the thickness of the spiral material,

$$\text{Maximum stress } \sigma_{max} = \frac{2T}{Z} = \frac{2T}{bt^2/6} = \frac{12T}{bt^2} \quad (11.12)$$

Example 11.23 A 5-mm wide and 0.3-mm thick flat spiral spring is 2 m long. The maximum stress is 600 MPa at the point of greatest bending moment. Determine the torque, the work stored and the number of turns to wind up the spring. E = 208 GPa.

Solution

Given

b = 5 mm, t = 0.3 mm

l = 2 m, σ_{max} = 600 MPa

E = 208 GPa

To find

— Torque
— Work stored
— Number of turns

Torque

Maximum stress, $\sigma_{max} = \frac{12T}{bt^2}$

or $$600 = \frac{12T}{5 \times 0.3^2} \quad \text{or} \quad T = 22.5 \text{ N} \cdot \text{mm}$$

Work stored

$$\theta = \frac{1.25Tl}{EI} = \frac{1.25 \times 22.5 \times 2000}{208\,000 \times (5 \times 0.3^3/12)} = 24 \text{ rad}$$

$$\text{Work stored} = \frac{1}{2}T\theta = \frac{1}{2} \times 22.5 \times 24 = 270 \text{ N} \cdot \text{mm}$$

Number of turns

Number of turns of the spindle = $\frac{24}{2\pi}$ = 3.82 turns

11.6 LEAF OR LAMINATED SPRINGS

Leaf or laminated springs are made up of a number of leaves of varying lengths but of equal width and thickness placed in laminations and loaded as a beam. These are widely used in carriages such as cars and railway wagons. There are two main types of leaf springs: semi-elliptic and quarter-elliptic or cantilever type.

Semi-elliptic Type

Let W = Central load
l = Length of span
b = Width of leaves
t = Thickness of leaves
y = Rise of crown (midpoint) above the level of ends

The leaves are put in the form of laminations one above the other. The contact will be throughout the surface if the radius of the upper leaf is increased by the thickness of the leaf underneath. However, the leaves are bent to the same radius R_o and thus the contact between the leaves takes place at their edges only as shown in Fig. 11.10a. In this way, each leaf is regarded as simply supported at the ends on the leaf below it, the load at the overhanging end being $W/2$ (Fig. 11.10b). Figure 11.10c shows the bending moment diagram for such loading. Over the central portion it is constant but at the ends it is linear.

From Eq. 9.1 for curved bars, $\frac{M}{EI} = \frac{1}{R'} - \frac{1}{R}$ (11.13)

where R and R' are the radii of curvature before and after bending.

As R is assumed to be constant initially, the radius of curvature R' in the strained condition must remain same for all leaves if it is to be ensured that the contact between them continues to remain at the ends only.

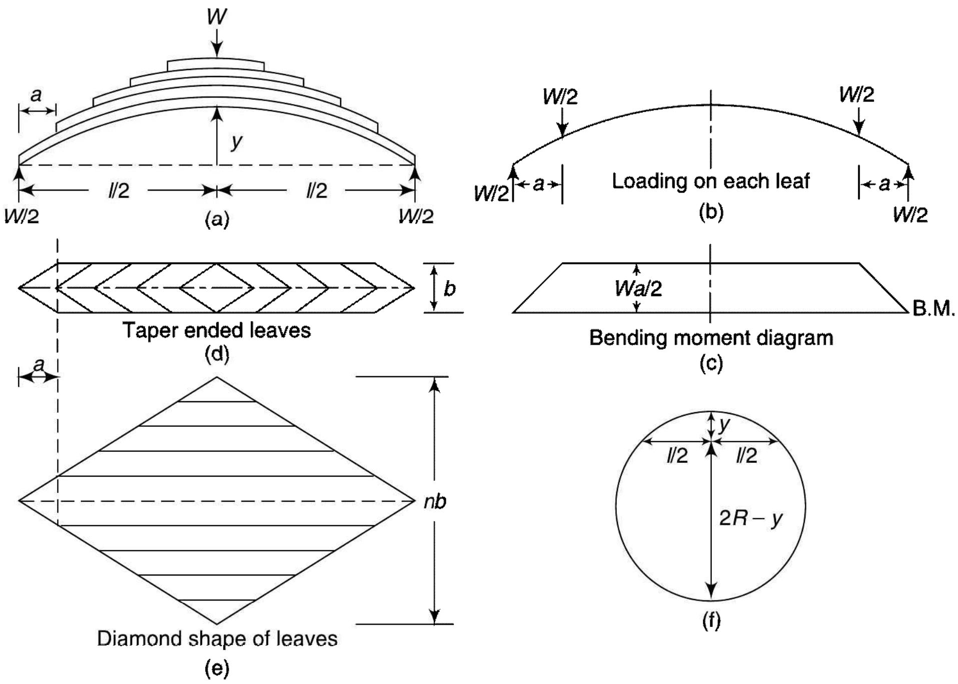

Fig. 11.10

This is possible if M/EI value in the above equation remains constant for all the leaves throughout the section. As M varies linearly at the ends, I must vary accordingly to fulfill the condition. This can be done by making the ends tapered uniformly to a point as shown in Fig. 11.10d. Thus to complete the set, the shortest leaf is made of a diamond shape. However, in practice the main leaf is not tapered to maintain full width where it is supported.

As each leaf is free to slide over the lower leaf and since the radius of curvature is the same for all the leaves, they can be assumed to be made of diamond shape by imagining that all the leaves except the main are cut longitudinally through the centre and placed as shown in Fig. 11.10e.

If n is the number of leaves, the overhang, $a = l/2n$

Width of the equivalent leaf = nb

Now, bending moment as well as the moment of inertia for the equivalent section are proportional to the distance from either end, the spring is equivalent to a beam of uniform strength, i.e., having the same stress throughout. Now, the analysis can be made by considering any section. Making use of the central section,

Bending moment, $M = -\dfrac{Wl}{4}$ tending to decrease the curvature

Moment of inertia, $I = \dfrac{nbt^3}{12}$

From the geometry of a circle (Fig. 11.10f), $y(2R - y) = \dfrac{l}{2} \cdot \dfrac{l}{2} = \dfrac{l^2}{4}$

Treating y to be small compared to R, $2yR = \dfrac{l^2}{4}$ or $\dfrac{1}{R} = \dfrac{8y}{l^2}$

Similarly, $\dfrac{1}{R'} = \dfrac{8y'}{l^2}$

Thus from Eq. 11.13, $\dfrac{-Wl/4}{E \cdot nbt^3/12} = \dfrac{8}{l^2}(y' - y)$

or deflection $\delta = y - y' = \dfrac{3}{8}\dfrac{Wl^3}{nbt^3E}$ (11.14)

The load which can straighten the spring ($y' = 0$) is called *Proof load.*

Proof load, $W = \dfrac{8nbt^3Ey}{3l^3}$ (11.15)

The maximum bending stress, $\sigma = \dfrac{M}{I} \cdot \dfrac{t}{2} = \dfrac{Wl/4}{nbt^3/12} \cdot \dfrac{t}{2} = \dfrac{3}{2}\dfrac{Wl}{nbt^2}$ (11.16)

Quarter-elliptic Type

The equivalent plan section is as shown in Fig. 11.11. The width varies from zero to nb at the fixed end. The treatment is similar as for semi-elliptic type. The results can be obtained directly also from the above results by replacing $2W$ for W and $2l$ for l.

$$\delta = y - y' = \frac{3}{8}\frac{(2W)(2l)^3}{nbt^3E} = \frac{6Wl^3}{nbt^3E} \quad (11.17)$$

and $$\sigma = \frac{3}{2}\frac{(2W)(2l)}{nbt^2} = \frac{6Wl}{nbt^2} \quad (11.18)$$

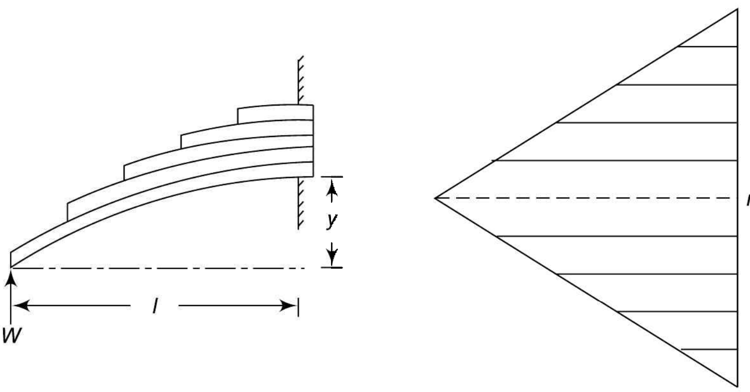

Fig. 11.11

Example 11.24 A leaf spring of semi-elliptical type having 8 plates is 1m long. The metallic plates have a proof stress in bending of 600 MPa. Each plate is 60 mm wide and 12 mm thick. Find the initial radius of the plates. Also find the height from which a load of 400 N may fall on the centre of the spring if the maximum stress so produced is one half of the proof stress. E = 204 GPa.

Solution

Given Semi-elliptical type leaf spring

$\sigma_p = 600$ MPa $l = 1$ m
$n = 8$ $b = 60$ mm
$t = 12$ mm $W = 400$ N
$E = 204$ GPa

To find

— Initial radius
— Height of drop

Initial radius

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R} - \frac{E}{R'} \qquad \text{...(Eq. 11.13)}$$

At proof load, R' is infinite and thus $E/R' = 0$

$$\therefore \quad \frac{\sigma}{y} = \frac{E}{R}$$

or $$\frac{600}{12/2} = \frac{204\,000}{R} \quad \text{or} \quad R = 2040 \text{ mm} \quad \text{or} \quad 2.04 \text{ m}$$

Determination of equivalent static load

Let W_e be the equivalent static load which produces the same maximum stress and deflection as the impact load.

Also, $$\sigma = \frac{3}{2}\frac{W_e l}{nbt^2}$$

As the maximum stress is one half of the proof stress,

$$\frac{600}{2} = \frac{3}{2} \cdot \frac{W_e \times 1000}{8 \times 60 \times 12^2} \quad \text{or} \quad W_e = 13\,824 \text{ N}$$

Height of drop

$$\delta = \frac{3}{8}\frac{W_e l^3}{nbt^3E} = \frac{3}{8} \times \frac{13\,824 \times 1000^3}{8 \times 60 \times 12^3 \times 204\,000} = 30.64 \text{ mm}$$

Now, $W(h+\delta) = \frac{1}{2}W_e \cdot \delta$

or $$400(h + 30.64) = \frac{1}{2} \times 13\,824 \times 30.64 \quad \text{or} \quad h = 498.8 \text{ mm}$$

Example 11.25 The length of the largest plate of a semi-elliptical laminated spring is 800 mm. The central load is 5.5 kN and the central deflection is 20 mm. The allowable bending stress is 200 MPa and the width of the plates is 10 times the thickness. Determine

(i) thickness and width of plates
(ii) number of plates
(iii) radius of curvature
(iv) overlap

Take E = 205 GPa.

Solution

Given Semi-elliptical type leaf spring

l = 800 mm	W = 5.5 kN
δ = 20 mm	$b = 10\,t$
σ = 200 MPa	E = 205 GPa

To find

— Thickness, width and number of plates
— radius of curvature
— overlap

Thickness and width

$$\delta = \frac{3}{8}\frac{Wl^3}{nbt^3E} = \frac{3}{2}\frac{Wl}{nbt^2} \cdot \frac{l^2}{4tE} = \sigma \cdot \frac{l^2}{4tE}$$

or $$20 = 200 \times \frac{800^2}{4t \times 205\,000}$$

or t = 7.8 mm and $b = 10 \times 7.8 = 78$ mm

Number of plates

$$\sigma = \frac{3}{2}\frac{Wl}{nbt^2}$$

or $$200 = \frac{3}{2} \cdot \frac{5500 \times 800}{n \times 78 \times 7.8^2} \quad \text{or} \quad n = 6.95, \text{ say } 7$$

Radius of curvature

Actual bending stress = $200 \times \dfrac{6.95}{7} = 198.6$ MPa

$$\frac{\sigma}{y} = \frac{E}{R} \quad \text{or} \quad R = \frac{205\,000 \times (7.8/2)}{198.6} = 4026 \text{ mm} \quad \text{or} \quad 4.026 \text{ m}$$

Overlap

$$a = \frac{l}{2n} = \frac{800}{2 \times 7} = 57.1 \text{ mm}$$

Example 11.26 The span of a simply supported and centrally loaded laminated steel spring is 650 mm. The central deflection of the spring does not exceed 40 mm for a proof load of 6 kN. The bending stress also does not exceed 360 MPa. Find the suitable values of width, thickness and the number of plates if they are available in multiples of 1 mm for thickness and 5 mm for width. Also determine the radius to which the plates should be formed. Assume the width to be ten times the thickness. E =205 GPa.

Solution

Given Laminated steel spring

$l = 650$ mm	$\delta = 40$ mm
$W = 6$ kN	$\sigma_b = 360$ MPa
$b = 10\,t$	E =205 GPa

Plates available in multiples of 1 mm for thickness and 5 mm for width.

To find

— thickness, width and number of plates

— radius of curvature

Thickness and width

$$\delta = \frac{3}{8}\frac{Wl^3}{nbt^3E} \quad \text{or} \quad 40 = \frac{3}{8}\cdot\frac{6000 \times 650^3}{n(10t)t^3 \times 205\,000} \quad \text{or} \quad nt^4 = 7535 \qquad \text{(i)}$$

Maximum stress, $$\sigma = \frac{3}{2}\frac{Wl}{nbt^3} \quad \text{or} \quad 360 = \frac{3}{2}\cdot\frac{6000 \times 650}{n(10t)t^3} \quad \text{or} \quad nt^3 = 1625 \qquad \text{(ii)}$$

From (i) and (ii), $t = \dfrac{7535}{1625} = 4.64$ mm

As the plates are available in multiples of 1 mm, let us take t =5 mm

Then width $b = 10 \times 5 = 50$ mm ...(50 is multiple of 5)

Number of plates

$$n = \frac{1625}{5^3} = 13$$

Radius of curvature

The actual deflection for a proof load of 6 kN,

$$\delta = \frac{3}{8}\cdot\frac{6000 \times 650^3}{13 \times 50 \times 5^3 \times 205\,000} = 37.1 \text{ mm}$$

At proof load, the spring is straight, therefore, the initial radius of curvature is

$$R = \frac{l^2}{8\delta} = \frac{650^2}{8 \times 37.1} = 1424 \text{ mm} \quad \text{or} \quad 1.424 \text{ m}$$

Example 11.27 A quarter-elliptic type of laminated spring is 800-mm long. The static deflection of the spring under an end load of 3 kN is 100 mm. Determine the number of leaves required and the maximum stress if the leaf is 75-mm wide and 8-mm thick. Also find the height from which a load can be dropped on to the undeflected spring to induce a maximum stress of 750 MPa. E = 208 GPa.

Solution

Given A quarter-elliptic type of laminated spring

$l = 800$ mm	$W = 3$ kN
$\delta = 100$ mm	b =75 mm
$t = 8$ mm	$\sigma = 750$ MPa
E =208 GPa	

To find

— Number of leaves
— maximum stress
— height of drop

Number of leaves

$$\delta = \frac{6Wl^3}{nbt^3E} \quad \text{or} \quad 100 = \frac{6 \times 3000 \times 800^3}{n \times 75 \times 8^3 \times 208\,000}; \; n = 11.54 \text{ say 12 leaves}$$

Maximum stress

$$\sigma = \frac{6Wl}{nbt^2} = \frac{6 \times 3000 \times 800}{12 \times 75 \times 8^2} = 250 \text{ N/mm}^2$$

Height of drop

Equivalent gradually applied load to induce a maximum stress of 750 MPa = $3000 \times \frac{750}{250} = 9000$ N

The corresponding deflection, $\delta = \frac{6 \times 9000 \times 800^3}{12 \times 75 \times 8^3 \times 208\,000} = 288.5$ mm

Loss of potential energy = Gain of strain energy

$$3000(h + 288.5) = \frac{1}{2} \times 9000 \times 288.5$$

Height from which the load can be dropped, $h = 432.8 - 288.5 = 144.3$ mm

Summary

1. Springs are used to absorb energy and to release the same according to the desired function to be performed or to absorb shocks or to deflect under external forces to provide the desired motion to a machine member.
2. Any member which can deform under a force can act as a spring. However, the term is normally used for those members that deform considerably under the action of forces without exceeding the safe limit of stresses.
3. *Stiffness* of a spring is defined as the force required for unit deflection.
4. Close-coiled helical springs are those in which the angle of helix is so small that the coils may be assumed to be in a horizontal plane if the axis of the spring is vertical.
5. When a close-coiled helical spring is acted upon by an axial load, there is axial extension and when an axial torque is applied, there is a change in the radius of curvature of the spring coils and there is angular rotation of the free end (wind-up).
6. In a close-coiled helical spring, deflection under an axial load, $\delta = \frac{8WD^3n}{Gd^4}$ and rotation about the axis under the action of an axial torque, $\varphi = \frac{64TDn}{Ed^4}$.
7. In an open-coiled helical spring acted upon by an axial load only,

$$\delta = \frac{8WD^2n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} - \frac{2\sin^2\alpha}{E}\right) \quad \text{and} \quad \varphi = \frac{16WD^2n\sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

Axial torque only,

$$\delta = \frac{32TDn}{d^4 \cdot \cos\alpha}\left(\frac{\sin^2\alpha}{G} - \frac{2\cos^2\alpha}{E}\right) \quad \text{and} \quad \varphi = \frac{16TD^2 n \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

8. A flat spiral spring consists of a uniform thin strip wound into a spiral in one plane and pinned at the outer end. It is wound up by applying a torque by turning the inner end at the centre of the spiral with a spindle. The wound spring is released slowly over a period of time.

   Rotation of the spiral spring, $\theta = \frac{1.25Tl}{EI}$ and Maximum stress $\sigma_{max} = \frac{12T}{bt^2}$

9. A leaf spring is made up of a number of leaves of varying length but of equal width and thickness placed in laminations and loaded as a beam. The leaves are bent to the same radius initially and the contact between the leaves takes place at their edges only.

10. In a semi-elliptic-type leaf spring, $\delta = y - y' = \frac{3}{8}\frac{Wl^3}{nbt^3E}$ and maximum bending stress, $\sigma = \frac{3}{2}\frac{Wl}{nbt^2}$

11. In a quarter-elliptic-type leaf spring, $\delta = \frac{6Wl^3}{nbt^3E}$ and $\sigma = \frac{6Wl}{nbt^2}$.

Objective Type Questions

1. The deflection of a closely coiled helical spring under an axial load is given by,

(a) $\frac{WR^3n}{Gr^4}$ (b) $\frac{2WR^3n}{Gr^4}$ (c) $\frac{4WR^3n}{Gr^4}$ (d) $\frac{8WR^3n}{Gr^4}$

2. Shear stress in a closed-coiled helical spring under an axial load is

(a) $\frac{8WD}{\pi d^3}$ (b) $\frac{4WD}{\pi d^3}$ (c) $\frac{8WD}{\pi d^2}$ (d) $\frac{16WD}{\pi d^3}$

3. The predominant effect of an axial tensile force on a helical spring is

(a) bending (b) tension (c) compression (d) twisting

4. The equivalent stiffness of two springs joined in series is

(a) $s = \frac{s_1 s_2}{s_1 + s_2}$ (b) $s = \frac{s_1 / s_2}{s_1 + s_2}$ (c) $s = s_1 + s_2$ (d) $s = s_1 \cdot s_2$

5. The equivalent stiffness of two springs joined in parallel is

(a) $s = \frac{s_1 s_2}{s_1 + s_2}$ (b) $s = \frac{s_1 / s_2}{s_1 + s_2}$ (c) $s = s_1 + s_2$ (d) $s = s_1 \cdot s_2$

6. The angle of twist of a closely coiled helical spring under an axial torque is

(a) $\frac{32TDn}{Ed^4}$ (b) $\frac{64TDn}{Ed^4}$ (c) $\frac{64Tdn}{ED^4}$ (d) $\frac{32Tdn}{ED^4}$

7. Rotation of a flat spiral spring is given by

(a) $\frac{Tl}{EI}$ (b) $\frac{1.5Tl}{EI}$ (c) $\frac{1.75Tl}{EI}$ (d) $\frac{1.25Tl}{EI}$

8. Maximum stress is a flat spiral spring is given by

(a) $\frac{12T}{bt^3}$ (b) $\frac{12T}{b^2t}$ (c) $\frac{12T}{bt^2}$ (d) $\frac{T}{12bt^2}$

9. Widely used springs in automobile industry are

(a) flat spiral springs (b) leaf springs

(c) closely-coiled helical springs (d) open-coiled helical springs

10. Proof load in a leaf spring is

(a) $\dfrac{8nbt^3Ey}{3l^3}$ (b) $\dfrac{3nbtEy}{8l^3}$ (c) $\dfrac{3nbt^3Ey}{8l}$ (d) $\dfrac{8nbt^2Ey}{3l^2}$

11. The central deflection of a leaf of leaf springs is

(a) $\dfrac{3}{8}\dfrac{Wl^2}{nbt^2E}$ (b) $\dfrac{8}{3}\dfrac{Wl^3}{nbt^3E}$ (c) $\dfrac{8}{3}\dfrac{Wl^2}{nbt^2E}$ (d) $\dfrac{3}{8}\dfrac{Wl^3}{nbt^3E}$

12. Leaf springs are subjected to

(a) tensile stress (b) compressive stress (c) shear stress (d) bending stress

Answers

1. (c)	2. (a)	3. (d)	4. (a)	5. (c)	6. (b)
7. (d)	8. (c)	9. (b)	10. (a)	11. (d)	12. (d)

Review Questions

11.1 What is a spring? What is its use?

11.2 How are the springs classified? Mention the use of each type.

11.3 What do you mean by a close-coiled helical spring? Deduce an expression for its deflection under the action of an axial load.

11.4 Show that a close-coiled helical spring does not rotate under axial load and does not deflect under torque.

11.5 Deduce an expression for the angle of rotation of one end relative to the other of a close-coiled helical spring under the action of an axial torque.

11.6 Deduce expression for the deformation of an open-coiled helical spring under the action of an axial load. Show that for small angle of helix, the expression reduces to that of a close-coiled spring.

11.7 Deduce expressions for the axial deflection and axial rotation of an open-coiled helical spring under the action of a torque.

11.8 What is a flat spiral spring? Deduce an expression for the rotation of the spiral.

11.9 What are leaf springs? What are their types? Develop relations to find the proof load and the maximum bending stress in case of semi-elliptic type leaf springs.

Numerical Problems

11.1 Determine the mean and the coil diameters of a close-coiled helical spring which has its mean diameter 10 times the wire diameter. The spring is to carry a load of 600 N and the shear stress in the material is to be 100 MPa. (124 mm, 12.4 mm)

11.2 A close-coiled spring consists of 12 coils of 12-mm diameter steel wire. The mean diameter of the spring is 140 mm. Find the spring constant. Also find the force required to elongate the spring by 50 mm. G = 80 GPa. (6.3 N/mm, 315 N)

11.3 The coil diameter of a close-coiled helical spring having 10 coils is eight times the wire diameter. The spring absorbs 60 N · m of energy when compressed by 40 mm. Find the coil and the wire diameters and the maximum shear stress. G = 85 GPa. (289 mm, 36.1 mm; 46.9 MPa)

11.4 A close-coiled helical spring having 12 coils of 16-mm wire diameter is subjected to an axial load of 320 N. The mean diameter of the coils is 240 mm. Determine the axial deflection, strain energy stored, maximum torsional shear stress and the maximum shear stress using Wahl's correction factor. G = 82 GPa. (79 mm, 12.64 N·m, 47.7 MPa, 52.2 MPa)

11.5 A close-coiled helical spring having eight coils has mean diameter of 75 mm and spring constant of 80 kN/m. Find the diameter of the spring wire if the maximum shear stress is not to exceed 260 MPa. Also find the maximum axial load which the spring can carry. G = 80 GPa. (12.82 mm; 2.868 N)

11.6 The mean coil diameter of a helical spring is 8 times the wire diameter. It is made to absorb 200 N·m of energy with a deflection of 100 mm. If the maximum shear stress is not to exceed 125 MPa, find mean diameter of the coils, wire diameter and the number of turns. G =77 GPa.

(204 mm, 25.5 mm, 12)

11.7 A close-coiled helical spring is made of 12-mm diameter steel wire. The spring has 12 turns and the mean diameter is 100 mm. Find the increase in the number of turns and the bending stress produced when the spring is subjected to an axial twist of 20 N · m. Also find the torsional stiffness of the spring. E =200 GPa. (0.059, 117.9 MPa; 54 N·m/rad).

11.8 A close-coiled helical spring has a stiffness of 900 N/m in compression with a maximum load of 45 N and a maximum shear stress of 120 MPa. The solid length of the spring is 42 mm. Determine the mean coil diameter, wire diameter and the number of coils. G = 40 GPa. (43 mm; 3.45 mm; 12.17)

11.9 Two concentric springs have a solid length of 50 mm. When they are subjected to an axial load of 6 kN, the maximum deflection is found to be 40 mm. The springs are made of the same material. The maximum permissible shear stress is 160 MPa. Determine the load shared by each spring and the wire diameters if the inner spring diameter is 120 mm and the radial clearance is 2.5 mm. G = 84 GPa.

(Outer: 3.373 kN, 2.627 kN; Inner: 5.89 mm, 5.2 mm)

11.10 The inner spring of a compound helical spring is arranged within and concentric with the outer one and is shorter by 9 mm. The outer spring has 10 coils with a mean diameter of 24 mm and a wire diameter of 3 mm. Determine the stiffness of the inner spring when an axial load of 150 N causes the outer spring to compress by 16 mm. If the radial clearance between the springs is 1.5 mm and the inner spring has 8 coils, find its wire diameter. G = 78 GPa. (6.51 N/mm; 2.15 mm)

11.11 A composite spring consists of two close-coiled springs in series and the stiffness of the composite spring is 1.3 N/mm. The mean diameter of each spring is eight times its wire diameter. One spring is of 2.5 mm diameter and has 20 coils. Determine the wire diameter of the other spring if it has 15 coils. Also find the maximum axial load which can be applied and the corresponding deflection for a maximum shear stress of 240 MPa. G =80 GPa. (2.136 mm, 53.8 N, 41.3 mm)

11.12 An open-coiled helical spring is made from a 6-mm diameter wire and has a mean diameter of 50 mm. If the spring is assumed to be closely coiled, it would induce an error of 2.5 % in axial deflection when loaded with axial load. Find the pitch of the open-coiled spring. Poisson's ratio is 0.3.

(48.6 mm)

11.13 A 2.5-m long flat spiral spring is made up of 2-mm thick and 8-mm wide wire. Assuming the permissible stress to be 540 MPa, find the torque, the work stored and the number of turns to wind up the spring. E = 210 GPa. (1.44 N·m, 2.89 N·m, 0.64)

11.14 A 600-mm long semi-elliptic spring is made of 10-mm thick and 42-mm wide steel plates. Find the number of plates required to carry a load of 5 kN if the induced stress is limited to 220 MPa. Also determine the central deflection and the initial radius of curvature under the load if the plates are straightened. E = 200 GPa. (5, 9.6 mm, 4.545 m)

11.15 A leaf spring of semi-elliptic type has 10 plates. Each plate is 10 mm thick and 80 mm wide. The length of the spring is 1.4 m. The material of the plates is steel having a proof stress in bending of 630 MPa. Find the initial radius to which the plates should be bent. Also find the height from which a load of 500 N can be dropped so that the maximum stress produced is half of the proof stress. E = 208 GPa.

(1.65 m; 816 mm)